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EE 5150–Transform Theory and Its Applications

Midterm Exam, Fall 2009

November 23th, 2009, 9:30-12:15 AM

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Q1) Determine the impulse response **(3p.)** and the step response of the following causal system **(3p.)**. Are the system stable? **(3p.)**

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

$$Y(z) = \frac{X(z)}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Impulse response: X(z)=1

$$Y(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u(n)$$

Step response: X(z)= $\frac{1}{1-z^{-1}}$

$$Y(z) = \frac{8/2}{1 - z^{-1}} - \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{1/3}{1 - \frac{1}{4}z^{-1}}$$

$$y(n) = \left[\frac{8}{3} - 2\left(\frac{1}{2}\right)^n - \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n)$$

Since the poles of H(z) are inside the unit circle, the system is stable (poles at z=$\frac{1}{2}, \frac{1}{4}$)

Q2) If X(z) is the z transform of x(n), show that **(8p.)**

$$Z\{x^*(n)\} = X^*(z^*)$$

$$\sum_{n=-\infty}^{+\infty} x^*(n)z^{-n} = \sum_{n=-\infty}^{+\infty} [x(n)(z^*)^{-n}]^* = X^*(z^*)$$

Q3) Determine thr causal signal x(n) if its z transform, X(z), is given by **(8p.)**

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$$

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-2z^{-1}}$$

Hence; A=2, B=-1.

$$x(n) = [2(-1)^n - (-2)^n]u(n)$$

Q4) Let x(n) be a sequence with z transform X(z). Determine, in terms of X(z), the z transform of the following signal. **(8p.)**

$$x_1(n) = x(2n)$$

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1(n)z^{-n} = \sum_{n=-\infty}^{+\infty} x_1(2n)z^{-n} = \sum_{k=-\infty}^{+\infty} x(k)z^{-k/2}$$

$= \sum_{k=-\infty}^{+\infty} \left[\frac{x(k)+(-1)^k x(k)}{2}\right] z^{-k/2}$; k even

$$= \frac{1}{2}\sum_{k=-\infty}^{+\infty} x(k)z^{-k/2} + \frac{1}{2}\sum_{k=-\infty}^{+\infty} x(k)\left(-z^{1/2}\right)^{-k}$$

$$= \frac{1}{2}\left[X(\sqrt{z}) + X(-\sqrt{z})\right]$$

Q5) Determine the signal x(n) if its Fourier transform is a given as **(6p.)**

$$x(n) = \frac{1}{2\pi}\left[\int_{\frac{8\pi}{10}}^{\frac{9\pi}{10}} e^{j\omega n}d\omega + \int_{\frac{-9\pi}{10}}^{\frac{-8\pi}{10}} e^{j\omega n}d\omega + 2\int_{\frac{9\pi}{10}}^{\pi} e^{j\omega n}d\omega + 2\int_{-\pi}^{\frac{-9\pi}{10}} e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{jn}\left(e^{\frac{jn9\pi}{10}} - e^{\frac{-jn9\pi}{10}} - e^{\frac{jn8\pi}{10}} + e^{\frac{-jn8\pi}{10}}\right) + \frac{2}{jn}\left(-e^{\frac{jn9\pi}{10}} + e^{\frac{-jn9\pi}{10}} + e^{jn\pi} - e^{-jn\pi}\right)\right]$$

$$= \frac{1}{n\pi}\left[\sin(\pi n) - \sin\left(\frac{8\pi n}{10}\right) - \sin\left(\frac{9\pi n}{10}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\sin\left(\frac{4\pi n}{5}\right) + \sin\left(\frac{9\pi n}{10}\right)\right]$$

Q6) Consider a signal $x(n)=\{-1, 2, \underbrace{3}_{\uparrow}, 2, -1\}$ with Fourier transform X(ω). Compute the following quantities, without explicity computing X(ω);

(a) X(0), **(2p.)** (b) Angle$\{X(\omega)\}$, **(2p.)** (c) X(π), **(2p.)** (d) $\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega$, **(2p.)** (e) $\int_{-\pi}^{\pi} X(\omega)d\omega$ **(2p.)**

(a) $x(0) = \sum_n x(n) = -1$

(b) $\text{angle}[X(\omega)] = \pi$ for all ω

(c) $x(\pi) = \sum_{n=-\infty}^{+\infty} x(n)e^{-jn\pi} = \sum_n (-1)^n x(n) = -3 - 4 - 2 = -9$

(d) $\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega = 2\omega \sum_n |x(n)|^2 = (2\pi)19 = 38\pi$

(e) $X(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)d\omega$

Then, $\int_{-\pi}^{\pi} X(\omega)d\omega = 2\pi X(0) = -6\pi$

Q7) Let $x_a(t)$ be an analog signal with bandwith B=3 kHz. We wish to use $N=L^m$-point DFT to compute the spectrum of the signal with a resolution less than or equal to 50 Hz. Determine the minimum sampling rate **(2p.)**, the minimum number of required samples **(2p.)** and the minimum length of the analog signal record **(2p.)**.

$F_1=F_N=2B=6000$ samples/sec.

$T=1/F_s=1/6000$

$$\frac{1}{LT} \leq 50 \Longrightarrow L \geq \frac{1}{50T} = \frac{6000}{50} = 120 \text{ samples}$$

$$LT = \frac{1}{6000} \cdot 120 = 0{,}02 \text{ sec.}$$

Q8) Compute the N-point DFTs of the following signal. **(8p.)**

$$x(n) = \cos\left(\frac{2\pi}{N}k_0 n\right), \qquad 0 \leq n \leq N-1$$

$$x(n) = \frac{1}{2}e^{j\frac{2\pi n}{N}k_0} + \frac{1}{2}e^{-j\frac{2\pi n}{N}k_0} = x_1(n) + x_2(n)$$

$$X_1(k) = \sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}k_0} e^{-j\frac{2k\pi n}{N}} = \sum_{n=0}^{N-1} e^{j\frac{-2\pi(k-k_0)n}{N}}$$

From this result

$$X(k) = \frac{N}{2}[\delta(k-k_0) + \delta(k-N+k_0)]$$

Q9) The DCT is defined as

$$\theta(k) = \frac{\sqrt{2}}{N}\sum_{n=0}^{N-1}\alpha(k)x(n)\cos\left[\frac{(2n+1)k\pi}{2N}\right] \qquad k=0, 1, \cdots, N-1$$

$$x(n) = \frac{\sqrt{2}}{N}\sum_{n=0}^{N-1}\alpha(k)\theta(k)\cos\left[\frac{(2n+1)k\pi}{2N}\right] \qquad k=0, 1, \cdots, N-1$$

where $\alpha(0) = \frac{1}{\sqrt{2}}$ and $\alpha(k) = 1;\ \ k \neq 0.$

For N=2, calculate the DCT **(2p.)**. What is the identical transform of this result included? **(4p.)**

$$C^2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ \cos(\pi/4) & \cos(3\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = L^2$$

The identical transform is Karhunen Loewe transform, L^2.

Q10) Consider a stationary source with variance of unity and normalized autocorrelation ρ between neighboring samples

$$R_{xx} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Find the eigenvalues, eigenvectors and the coefficients of KLT. **(10p.)**

$$\begin{vmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{vmatrix} = 0 \qquad \lambda_0 = 1+\rho, \quad \lambda_1 = 1-\rho$$

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} I_0(0) \\ I_0(1) \end{bmatrix} = (1+\rho)\begin{bmatrix} I_0(0) \\ I_0(1) \end{bmatrix}, \quad I_0(0) = -I_0(1)$$

Similarly, for I_1, find that $I_1(0)=-I_1(1)$

If we scale the vectors, so that

$$I_0^2(0) + I_0^2(1) = I_1^2(0) + I_1^2(1) = 1$$

The above equation show that the product of the eigenvalues λ_0 and λ_1 equals.

$\prod_{i=1}^{N-1} \lambda_i = \lambda_0\lambda_1 = 1 - \rho^2|R_{xx}|$ and

$R_{\theta\theta} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}$, hovewer l_0 and l_1 are the row of L and for orthogonality, LL^T=I.

Therefore

$$L^2 = (L^T)^2 = (L^{-1})^2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Q11) Find the Laplace transform of f(t). **(6p.)**

$$f(t) = e^{-2t}\cos(\sqrt{3}) - t^2e^{-2t}$$

$$F(s) = L\{e^{-2t}\cos(\sqrt{3}) - t^2e^{-2t}\} = L\{e^{-2t}\cos(\sqrt{3})\} - L\{t^2e^{-2t}\} = \frac{3+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}$$

Q12) Laplace transforms are useful for solving linear differential equations that arise in many settings. Consider a linearized model of a motor with a torque control system, for example, where the actual torque of the motor relative to a desired torque is described by the following Laplace transform.

$$H(s) = \frac{\frac{k_p}{L}\left(s + \frac{k_I}{k_p}\right)}{s^2 + \frac{k_p}{L}s + \frac{k_I}{L}}$$

where L=1 mH=inductance of motor windings, k_p=gain for proportional feedback, k_I=gain for integral feedback. The inverse Laplace transform of H(s) contains multiplicative factors of the form e^{-at} that determine how fast time waveforms decay to zero. This decay corresponds to how rapidly the motor torque reaches a desired value. A faster decay is desirable. Equal roots, (i.e. critical damping), is optimal if vibration (oscillatory solutions) must be eliminated. Find the inverse Laplace transform of H(s) for the values; a=1.6k, k_I=2.56G. **(10p.)**

$$H(s) = \frac{\frac{k_p}{L}\left(s + \frac{k_I}{k_p}\right)}{s^2 + \frac{k_p}{L}s + \frac{k_I}{L}} = \frac{\frac{a}{2}\left(s + \frac{a}{2}\right)}{(s+a)^2}$$

We can factor out an s+a fron the top, as follows.

$$H(s)=\frac{\frac{a}{2}\left(s+a-\frac{a}{2}\right)}{(s+a)^2}=\frac{a}{2}\left(\frac{1}{s+a}-\frac{a}{2}\cdot\frac{1}{(s+a)^2}\right)$$

The inverse Laplace transform now follows.

$$h(t)=\frac{a}{2}\left(e^{-at}-\frac{a}{2}te^{-at}\right)=\frac{a}{2}e^{-at}\left(1-\frac{a}{2}t\right)$$

Using the value of a, we have the numerical answer.

$h(t)=800e^{-800t}(1-800t)=800e^{-t/1.25ms}(1-t/1.25ms)$

Q13) Find the Fourier sine transform of $e^{-x},\ x\geq 0$ **(2p.)**. By using this result, show that **(3p.)**

$$\int_0^\infty \frac{x\sin(mx)}{x^2+1}dx=\frac{\pi}{2}e^{-m},\qquad m>0$$

$$\frac{\sqrt{2}}{\pi}\left[\frac{\alpha}{(1+\alpha^2)}\right]$$