FOURIER SERIES

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April, 12-16: Wendesday 3-4, Thursday 3-4

- What is the meaning of the Fourier Series?

- Fourier series and Fourier transform are useful a mathematical tool for signal analysis.

- Fourier series were founded by Joseph Fourier when he was investigated heat variations on a circular metal.

- If you want to take Fourier series expansion of a signal, your signal must provide some conditions.

What is the conditions of Fourier series expansion for a signal?

Answer:

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Cond. 1: A function f(t) must be absolutely integrable over a single period T. So,

$$\int_T |f(t)|dt < \infty$$

Cond. 2: The number of local maxima and minima of f(t) is finite for any finite period of time.

Cond. 3: There is a finite number of discontinuities in the function f(t) for any finite period of time.

Fourier Series

- Trigonometric Fourier series
- Exponential Fourier series

Trigonometric Fourier Series

If a signal has a periodic waveform, it can be explained as a series of harmonically related sinusoids. It has a fundamentally frequency or first harmonic.

Conventionally, a periodic signal f(t) as trigonometric Fourier series, can be expressed as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

The first term is a constant and represents DC component of the signal.

- Now, we calculate the Fourier series coefficients using trigonometric Fourier series expansion. To find coefficients, we will use the orthogonality property.

$$\int_0^{2\pi} \sin(mt)\,dt = 0 \qquad \int_0^{2\pi} \cos(mt)\,dt = 0$$

From these equations;

$$\int_0^{2\pi} \sin(mt)\cos(nt)\,dt = 0$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

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$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

If m is equal to n, then

$$\int_0^{2\pi} [\sin(mt)]^2\,dt = \int_0^{2\pi} [\cos(mt)]^2 dt = \pi$$

For simplicity, we assume that ω=1. Then

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \cdots + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots$$

To derive coefficients from this way, we multiply both sides with sin2t.

$$f(t) \sin 2t = \frac{a_0}{2} \sin 2t + a_1 \cos t \sin 2t + \cdots + b_1 \sin t \sin 2t + b_2 \sin 2t \sin 2t + \cdots$$

After that multiplying terms of both sides by ‘dt’, we integrate the period from 0 to 2π.

$$\int_0^{2\pi} f(t) \sin 2t \, dt = \frac{a_0}{2} \int_0^{2\pi} \sin 2t \, dt + a_1 \int_0^{2\pi} \cos t \sin 2t \, dt + \cdots + b_1 \int_0^{2\pi} \sin t \sin 2t \, dt + b_2 \int_0^{2\pi} (\sin 2t)^2 \, dt + \cdots$$

We observe that the terms of sin multiplying cos, sin multiplying sin and cos multiplying cos are equal to zero from orthogonality in the right side of the above equation. Only, the one term is not equal to zero,

$$b_2 \int_0^{2\pi} (\sin 2t)^2 \, dt$$

Then

$$\int_0^{2\pi} f(t) \sin 2t \, dt = b_2 \int_0^{2\pi} (\sin 2t)^2 \, dt = b_2 \pi$$

$$b_2 = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin 2t \, dt$$

The coefficients a_0, a_n and b_n

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$

Symmetry in Trigonometric Fourier Series

Types of Symmetry

- **Odd Symmetry**; If a signal is an odd fuction, it will include sine terms, only. That is, a_0 and a_n will be zero.

  Odd function; f(-t)=-f(t)
- **Even Symmetry**; If a signal is an even fuction, it will include cosine terms, only. That is, b_n will be zero.

  Even function; f(-t)=f(t)
- **Half-wave Symmetry**; If a signal has an half wave symmetry, only odd (odd cosine and odd sine) harmonics will exist. Other harmonics will be zero.

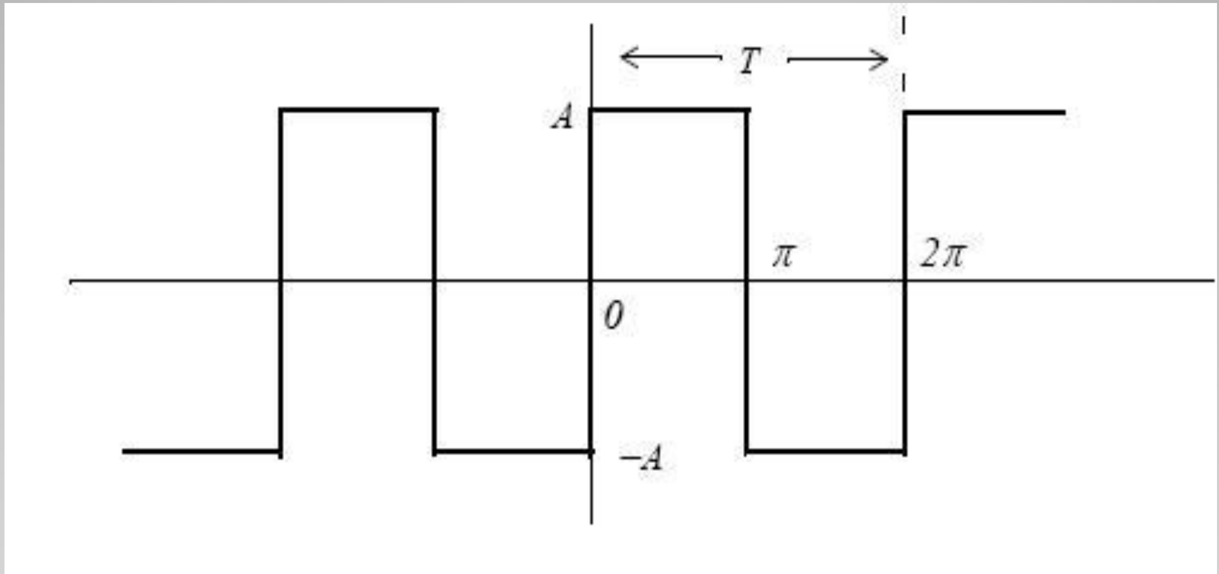
Half-Wave Symmetry

- Any periodic signal with period T is expressed matematically as $f(t)=f(t+T)$.
- If it has half-wave symmetry, it will illustrate by $f(t)= -f(t+T/2)$.
- The meaning of the above equation;

  the shape of the negative cycle of the signal is the same as the positive half cycle.

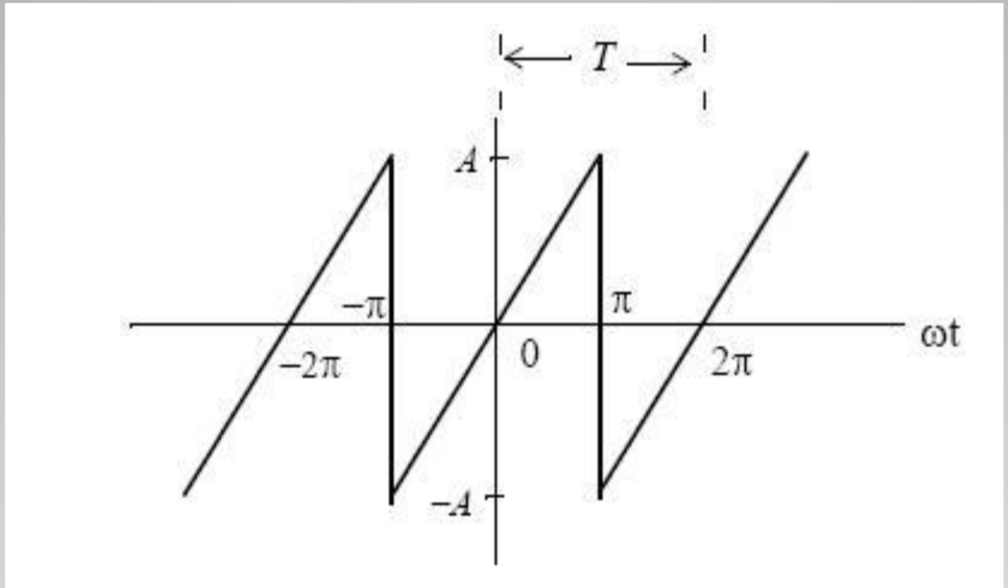
Examples

FS.1.



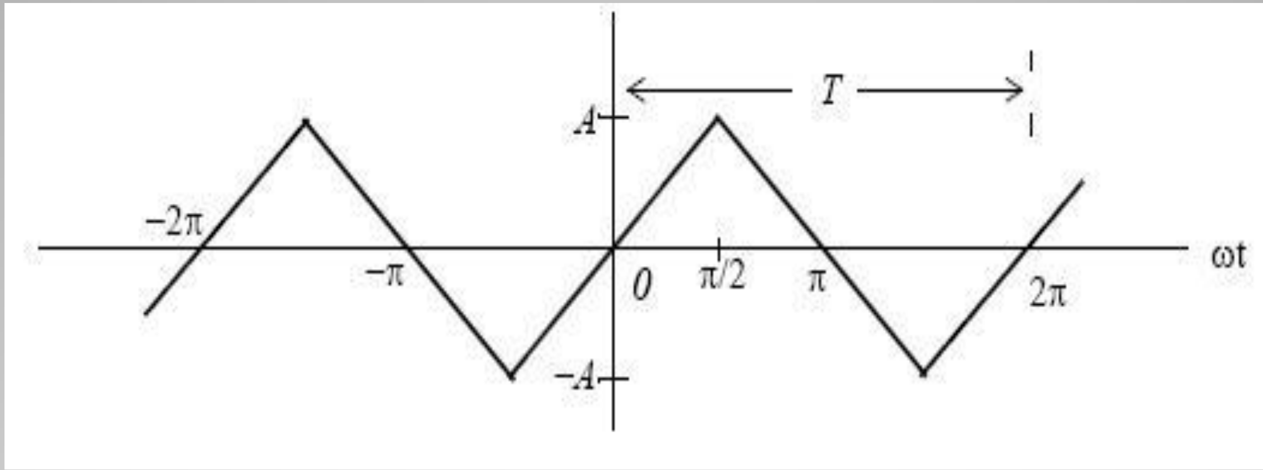
$$f(t) = \begin{cases} A, & 0 < t < T/2 \\ -A, & T/2 < t < T \end{cases}$$

FS.2.



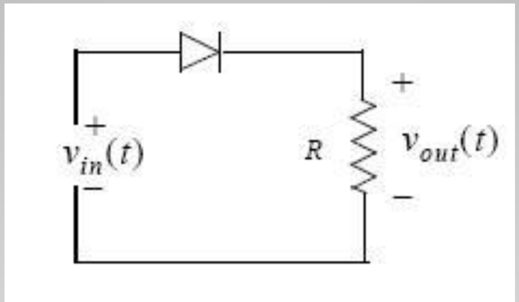
$$f(t) = \begin{cases} \dfrac{At}{\pi}, & 0 < t < \pi \\ \dfrac{At}{\pi} - 2A, & \pi < t < 2\pi \end{cases}$$

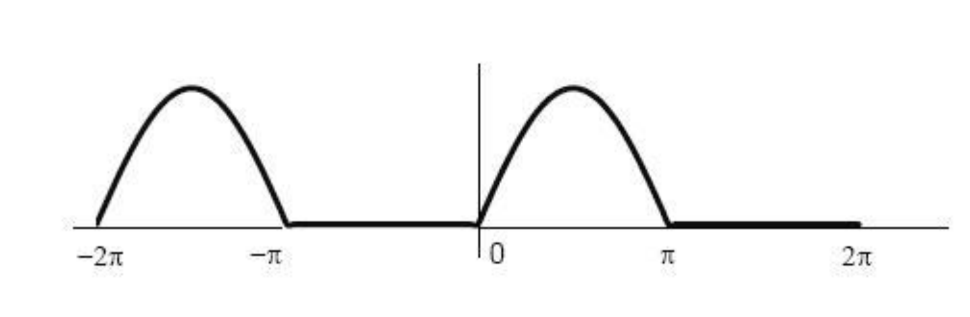
FS.3.



$$\int x \sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$$

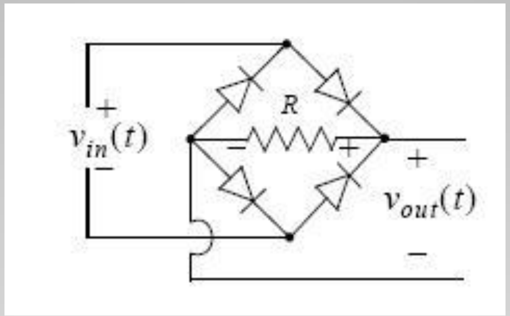
FS.4. Half-wave rectifier

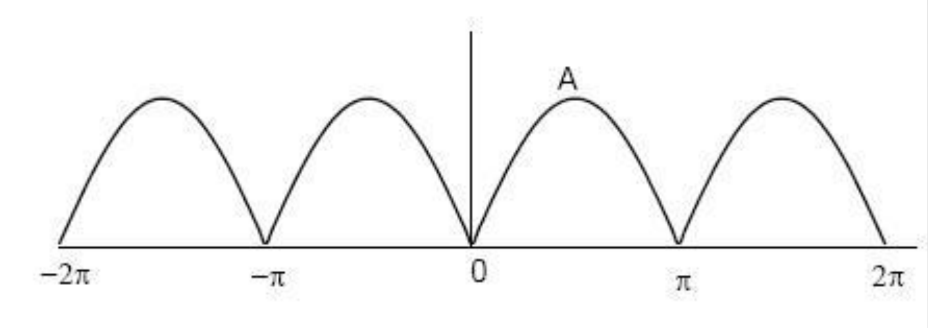




$$V_{out}(t) = \begin{cases} \sin(\omega t) & 0 < \omega t < \pi \\ 0 & \pi < \omega t < 2\pi \end{cases}$$

FS.5. Full-wave rectifier





$$V_{in}(t) = A\sin(\omega t)$$
$$V_{out}(t) = |A\sin(\omega t)|$$

Exponential Fourier Series

- A signal is expressed in an exponential form.
- The advantage of the exponential Fourier series with respect to the trigonometric Fourier series is to need less integration. We must calculate three integration for coefficients a_0, a_n and b_n in trigonometric Fourier series. However in exponentional form, we must take only one integration.

Exponential form is derived from the trigonometric form for the signal f(t) as follows:

- Substituting terms of

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

  into f(t).

$$f(t) = \frac{a_0}{2} + a_1\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) + a_2\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) + \cdots + b_1\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) + b_2\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) + \cdots$$

- Grouping terms with same exponents

$$f(t) = \cdots + \left(\frac{a_2}{2} - \frac{b_2}{2j}\right)e^{-2j\omega t} + \left(\frac{a_1}{2} - \frac{b_1}{2j}\right)e^{-j\omega t} + \frac{a_0}{2} + \left(\frac{a_1}{2} + \frac{b_1}{2j}\right)e^{j\omega t} + \left(\frac{a_2}{2} - \frac{b_2}{2j}\right)e^{2j\omega t} + \cdots$$

- Terms in paranthesis are represented by

$$C_{-n} = \frac{1}{2}\left(a_n - \frac{b_n}{j}\right) = \frac{1}{2}(a_n + jb_n)$$

$$C_n = \frac{1}{2}\left(a_n + \frac{b_n}{j}\right) = \frac{1}{2}(a_n - jb_n) \qquad C_0 = \frac{a_0}{2}$$

$$f(t) = \cdots + C_{-2}e^{-2j\omega t} + C_{-1}e^{-j\omega t} + \frac{C_0}{2} + C_1 e^{j\omega t} + C_2 e^{2j\omega t} + \cdots$$

- Remember that C_n coefficents are coplex.

$$C_{-n} = C_n^*$$

- Multiplying both sides of f(t) with $e^{-jn\omega t}$ and integrating over one period.

  For simplicity, we assume that ω=1.

$$\int_0^{2\pi} f(t)e^{-jnt}dt = \cdots$$

$$+\int_0^{2\pi} C_{-2}e^{-j2t}e^{-jnt}dt + \int_0^{2\pi} C_{-1}e^{-jt}e^{-jnt}dt + \int_0^{2\pi} C_0 e^{-jnt}dt + \int_0^{2\pi} C_1 e^{jt}e^{-jnt}dt$$

$$+\int_0^{2\pi} C_2 e^{j2t}e^{-jnt}dt + \cdots + \int_0^{2\pi} C_n e^{jnt}e^{-jnt}dt + \cdots$$

- Because of $\displaystyle\int_0^{2\pi} C_2 e^{j2t}e^{-jnt}dt = \int_0^{2\pi} -C_{-2}e^{-j2t}e^{-jnt}dt$

  We observe that the right side of the above equation are zero out of the last term. Then

$$\int_0^{2\pi} f(t)e^{-jnt}dt = \int_0^{2\pi} C_n e^{jnt}e^{-jnt}dt = \int_0^{2\pi} C_n dt = 2\pi C_n$$

In general, $\omega \neq 1$.

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega t}dt \quad \text{or} \quad C_n = \frac{1}{T}\int_0^{T} f(t)e^{-jn\omega t}dt$$

We can derive coefficients of the trigonometric Fourier series from coefficients of the exponentential Fourier series.

$$C_n + C_{-n} = \frac{1}{2}(a_n - jb_n + a_n + jb_n)$$

$$a_n = C_n + C_{-n}$$

$$C_n - C_{-n} = \frac{1}{2}(a_n - jb_n - a_n - jb_n)$$

$$b_n = j(C_n + C_{-n})$$

Symmetry in Exponential Fourier Series

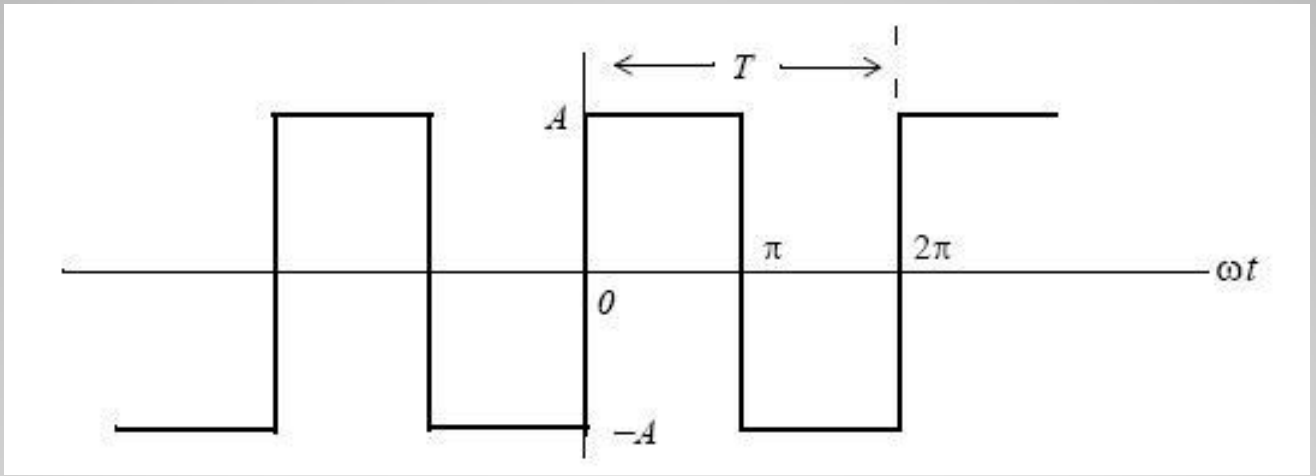
- For odd function, all coefficient are imaginary. Since even functions have only cosine terms.

  Odd function; f(-t)=-f(t)
- For even function, all coefficient are real. Since even functions have no cosine terms, only sine terms.

  Even function; f(-t)=f(t)
- If there is a half-wave symmetry, C_n=0 for n=even.
- If there is no symmetry, the signal f(t) is complex.

Examples

FS.6.



Examples

FS.7.