FOURIER TRANSFORM

INSTRUCTOR: DR. GÜLDEN KÖKTÜRK

If a signal is not periodic, it is expand with

FOURIER TRANSFORM

$$\int_{-\infty}^{+\infty} |f(t)|dt < \infty$$

i.e.; absolutely summable

Fourier transform or Fourier integral of a signal f(t)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

The inverse Fourier transform is defined as,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega$$

Mathematical notation of the Fourier transform and the inverse Fourier transform is,

$$\mathcal{F}\{f(t)\} = F(\omega) \qquad \mathcal{F}^{-1}\{F(\omega)\} = f(t)$$

Generally, the Fourier transform is complex. Therefore, it represents a sum of real and imaginary parts.

$$F(\omega) = \mathrm{Re}\{F(\omega)\} + j\mathrm{Im}\{F(\omega)\} = |F(\omega)|e^{j\varphi(\omega)}$$

Special Forms of the Fourier Transform

- If the signal f(t) is complex, then it can expressed as a sum of the real and imaginary parts of f(t).

$$f(t) = f_{Re}(t) + jf_{Im}(t)$$

With substituting above equation in Fourier integral, we obtain

$$F(\omega) = \int_{-\infty}^{+\infty} f_{Re}(t)e^{-j\omega t}dt + j\int_{-\infty}^{+\infty} f_{Im}(t)e^{-j\omega t}dt$$

From Euler's identity

$$F(\omega) = \int_{-\infty}^{+\infty} [f_{Re}(t)\cos\omega t + f_{Im}(t)\sin\omega t]dt + j\int_{-\infty}^{+\infty} [f_{Re}(t)\sin\omega t - f_{Im}(t)\cos\omega t]dt$$

The real and imaginary parts of F(ω) are,

$$F_{Re}(\omega) = \int_{-\infty}^{+\infty} [f_{Re}(t)\cos\omega t + f_{Im}(t)\sin\omega t]dt$$

$$F_{Im}(\omega) = -\int_{-\infty}^{+\infty} [f_{Re}(t)\sin\omega t - f_{Im}(t)\cos\omega t]dt$$

Similarly, the inverse Fourier transform is denoted by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [F_{Re}(\omega) + jF_{Im}(\omega)]e^{j\omega t}dt$$

From Euler's identity again

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[F_{Re}(\omega)\cos\omega t - F_{Im}(\omega)\sin\omega t]dt + \frac{j}{2\pi}\int_{-\infty}^{+\infty}[F_{Re}(\omega)\sin\omega t + F_{Im}(\omega)\cos\omega t]dt$$

The real and imaginary parts of the inverse Fourier transform are

$$f_{Re}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[F_{Re}(\omega)\cos\omega t - F_{Im}(\omega)\sin\omega t]dt$$

$$f_{Im}(t) = \frac{j}{2\pi}\int_{-\infty}^{+\infty}[F_{Re}(\omega)\sin\omega t + F_{Im}(\omega)\cos\omega t]dt$$

- **Real time functions;** if f(t) is real, that is

$$F_{Re}(\omega) = \int_{-\infty}^{+\infty} f_{Re}(t)\cos\omega t\,dt$$

$$F_{Im}(\omega) = -\int_{-\infty}^{+\infty} f_{Re}(t)\sin\omega t\,dt$$

F(ω) is complex.

If $f_{Re}(t)$ is even, that is $f_{Re}(-t) = f_{Re}(t)$

$$\left.\begin{aligned} F_{Re}(\omega) &= 2\int_{0}^{+\infty} f_{Re}(t)\cos\omega t\,dt \\ F_{Im}(\omega) &= -\int_{-\infty}^{+\infty} f_{Re}(t)\sin\omega t\,dt = 0 \end{aligned}\right\} \quad f_{Re}(t) = \text{even}$$

Finally, if f(t) is real and even, F(ω) is also real and even.

If $f_{Re}(t)$ is odd, that is $-f_{Re}(-t)= f_{Re}(t)$

$$\left.\begin{aligned} F_{Re}(\omega) &= \int_{-\infty}^{+\infty} f_{Re}(t)\cos\omega t\,dt = 0 \\ F_{Im}(\omega) &= -2\int_{0}^{+\infty} f_{Re}(t)\sin\omega t\,dt \end{aligned}\right\} \quad f_{Re}(t) = \text{odd}$$

Finally, if f(t) is real and odd, F(ω) is imaginary and odd.

- **Imaginary time functions;** if f(t) is imaginary,

$$F_{Re}(\omega) = \int_{-\infty}^{+\infty} f_{Im}(t)\sin\omega t\,dt$$

$$F_{Im}(\omega) = \int_{-\infty}^{+\infty} f_{Im}(t)\cos\omega t\,dt$$

F(ω) is complex.

If $f_{Im}(t)$ is even, that is $f_{Im}(-t)= f_{Im}(t)$

$$\left.\begin{aligned} F_{Re}(\omega) &= \int_{-\infty}^{+\infty} f_{Im}(t)\sin\omega t\,dt = 0 \\ F_{Im}(\omega) &= 2\int_{0}^{+\infty} f_{Im}(t)\cos\omega t\,dt \end{aligned}\right\} \qquad f_{Im}(t) = \text{even}$$

Finally, if f(t) is imaginary and even, F(ω) is also is imaginary and even.

If $f_{Im}(t)$ is odd, that is $-f_{Im}(-t)= f_{Im}(t)$

$$\left.\begin{aligned} F_{Re}(\omega) &= \int_{-\infty}^{+\infty} f_{Im}(t)\sin\omega t\,dt = 2\int_{0}^{+\infty} f_{Im}(t)\sin\omega t\,dt \\ F_{Im}(\omega) &= \int_{-\infty}^{+\infty} f_{Im}(t)\cos\omega t\,dt = 0 \end{aligned}\right\} \quad f_{Im}(t) = \text{odd}$$

Finally, if f(t) is imaginary and odd, F(ω) is real and odd.

$$F(-\omega) = F^{*}(\omega)$$

f(t)=Real

Time and Frequency Domain Relationship

f(t)	F(ω)				
	Real	Imaginary	Complex	Even	Odd
Real			✓		
Real and Even	✓			✓	
Real and Odd		✓			✓
Imaginary			✓		
Imaginary and Even		✓		✓	
Imaginary and Odd	✓				✓

Properties of the Fourier Transform

- **Linearity;** If $F_1(\omega)$ is the Fourier transform of $f_1(t)$, If $F_2(\omega)$ is the Fourier transform of $f_2(t)$, and so on, the linearity of the Fourier transform shows that

$$a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t) \Leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) + \cdots + a_n F_n(\omega)$$

- **Symmetry;** If $F(\omega)$ is the Fourier transform of $f(t)$, the symmetry of the Fourier transform shows that

$$F(t) \Leftrightarrow 2\pi f(-\omega)$$

- **Time Scaling;** If $F(\omega)$ is the Fourier transform of $f(t)$ and ‘a’ is real constant, then

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

- **Time Shifting;** If F(ω) is the Fourier transform of f(t), then

$$f(t-t_0) \Leftrightarrow F(\omega)e^{-j\omega t_0}$$

- **Frequency Shifting;** If F(ω) is the Fourier transform of f(t), then

$$e^{j\omega_0 t}f(t) \Leftrightarrow F(\omega-\omega_0)$$

$$e^{j\omega_0 t}f(at) \Leftrightarrow \frac{1}{|a|}F\left(\frac{\omega-\omega_0}{a}\right)$$

$$f(t)\cos\omega_0 t \Leftrightarrow \frac{F(\omega-\omega_0)+F(\omega+\omega_0)}{2}$$

$$f(t)\sin\omega_0 t \Leftrightarrow \frac{F(\omega-\omega_0)-F(\omega+\omega_0)}{2j}$$

- **Time Differentiation;** If F(ω) is the Fourier transform of f(t)

$$\frac{d^n f(t)}{dt^n} \Leftrightarrow (j\omega)^n F(\omega)$$

- **Frequency Differentiation;** If F(ω) is the Fourier transform of f(t)

$$(-jt)^n f(t) \Leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

- **Time Integration;** If F(ω) is the Fourier transform of f(t)

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

- **Conjugate Time and Frequency Functions;** If F(ω) is the Fourier transform of complex function f(t)

$$f^*(t) \Leftrightarrow F^*(-\omega)$$

- **Time Convolution;** If $F_1(\omega)$ is the Fourier transform of $f_1(t)$, If $F_2(\omega)$ is the Fourier transform of $f_2(t)$

$$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega) \cdot F_2(\omega)$$

- **Frequency Convolution;** If $F_1(\omega)$ is the Fourier transform of $f_1(t)$, If $F_2(\omega)$ is the Fourier transform of $f_2(t)$, then

$$f_1(t) \cdot f_2(t) \Leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

- **Area Under f(t);** If F(ω) is the Fourier transform of complex function f(t)

$$F(0) = \int_{-\infty}^{\infty} f(t)dt$$

- **Area Under F(ω);** If F(ω) is the Fourier transform of complex function f(t)

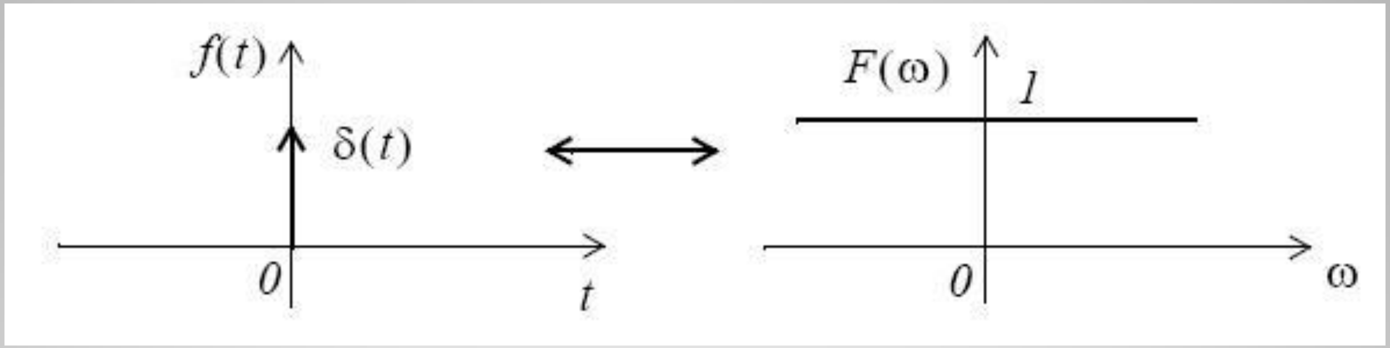
$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)d\omega$$

- **Parseval's Theorem;** If F(ω) is the Fourier transform of complex function f(t), the Parseval's relationship is denoted by

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Fourier Transform of Special Functions

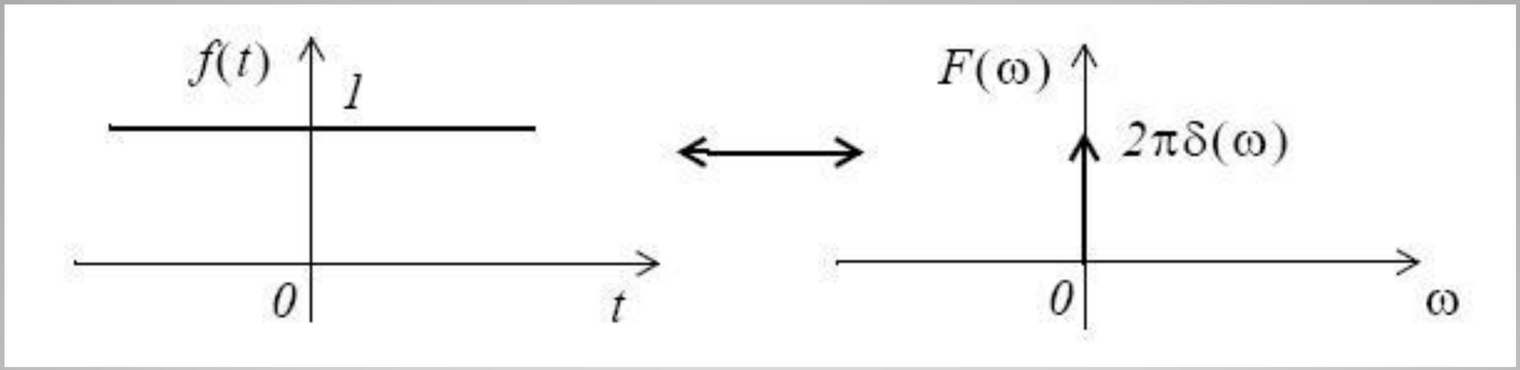
- **Delta Function;** $\delta(t) \Leftrightarrow 1$



$$\delta(t - t_o) \Leftrightarrow e^{-j\omega t_o}$$

- **Constant Function;**

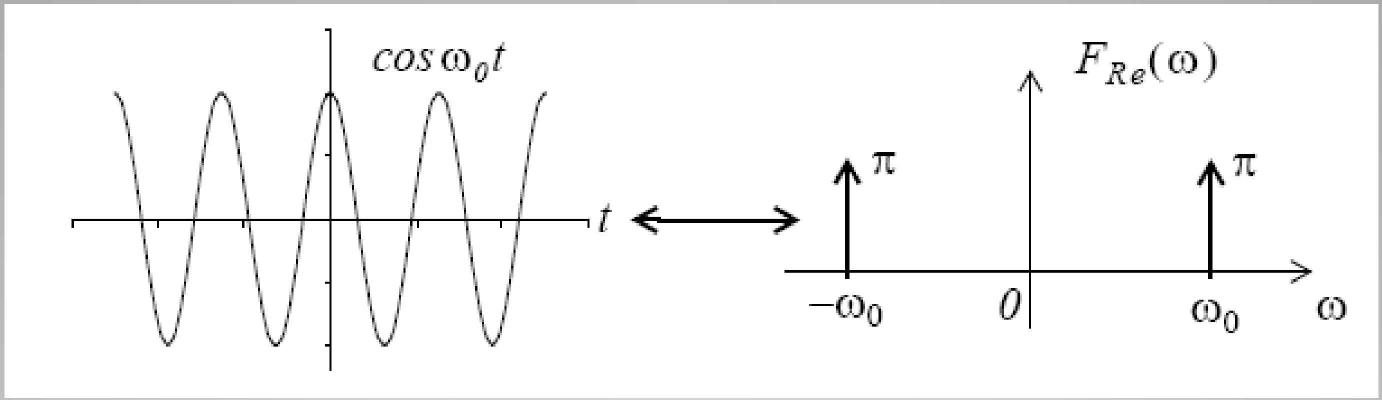
$$A \Leftrightarrow 2A\pi\delta(\omega)$$



$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$$

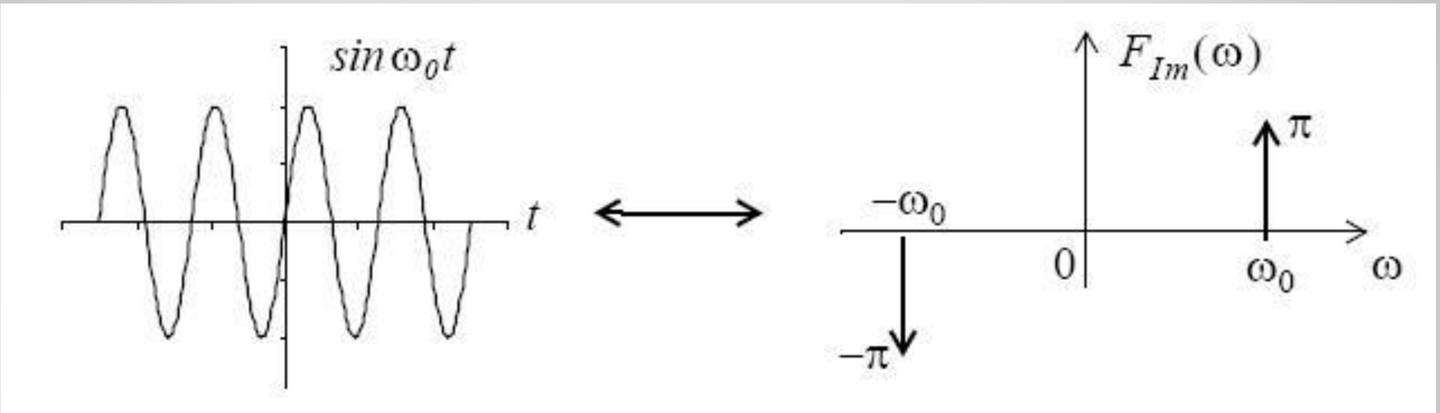
- **Cosine Function;**

$$\cos\omega_o t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \Leftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



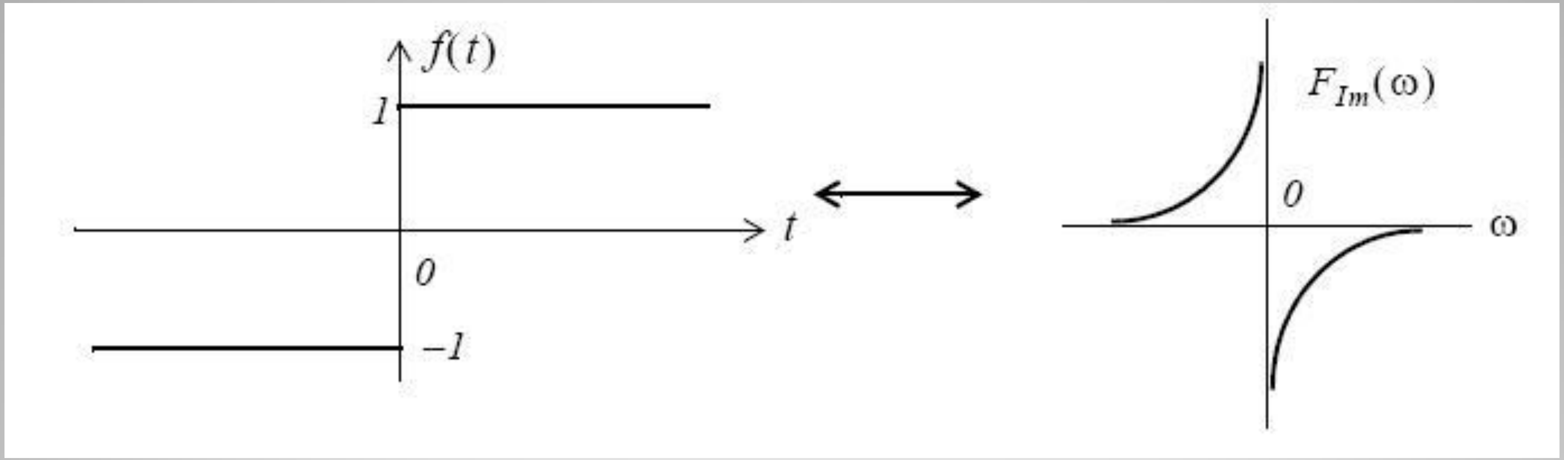
- **Sine Function;**

$$\sin\omega_o t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Leftrightarrow j\pi\delta(\omega-\omega_0) - j\pi\delta(\omega+\omega_0)$$



- **Signum Function;**

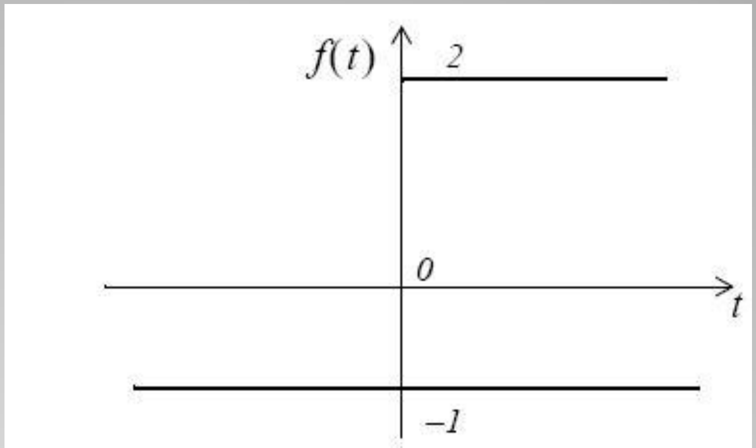
$$\mathbf{sgn(t) = u_0(t) - u_0(-t) \Leftrightarrow \frac{2}{j\omega}}$$

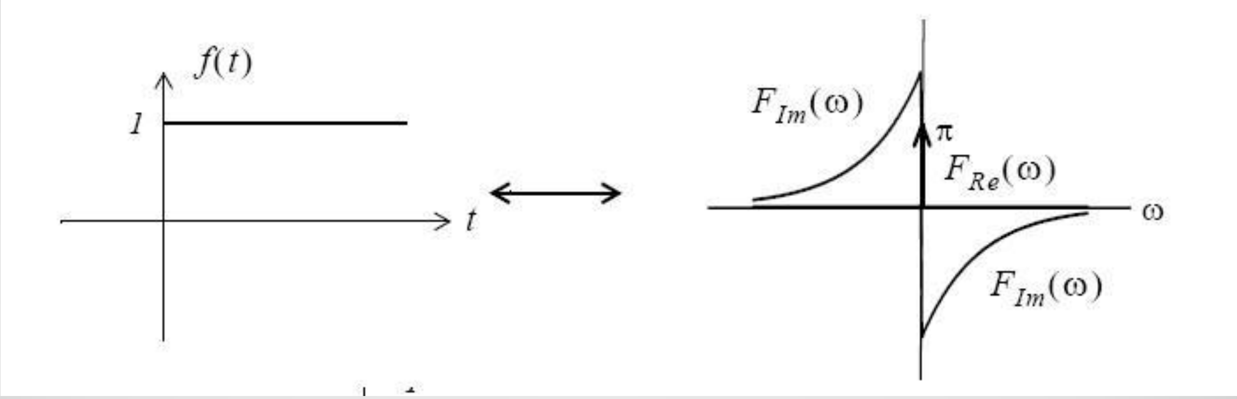


- **Unit Step Function;**

$$\mathbf{u_0(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\mathbf{sgn(t) = 2u_0(t) - 1}$$





- $\mathbf{e^{-j\omega_0 t}u_0(t)}$ **Function;**

$$\mathbf{e^{-j\omega_0 t}u_0(t) \Leftrightarrow 2\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}}$$

- $\cos(\omega_o t)u_0(t)$ **Function;**

$$\cos(\omega_o t)u_0(t) \Leftrightarrow \frac{\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \frac{1}{2j(\omega-\omega_0)} + \frac{1}{2j(\omega+\omega_0)}$$
$$\Leftrightarrow \frac{\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \frac{j}{{\omega_0}^2-\omega^2}$$

- $\sin(\omega_o t)u_0(t)$ **Function;**

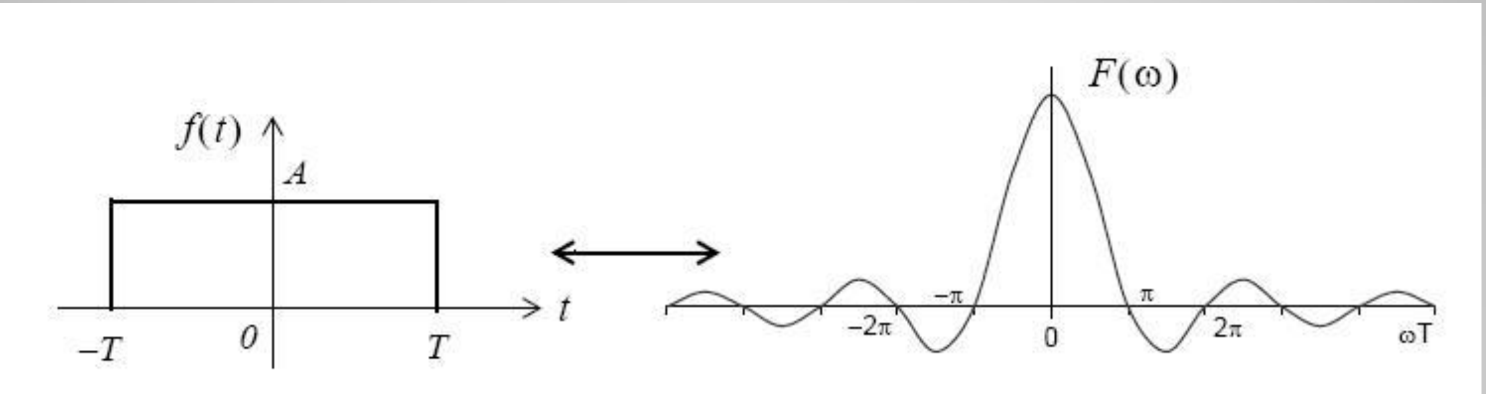
$$\sin(\omega_o t)u_0(t) \Leftrightarrow \frac{\pi}{2j}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \frac{\omega^2}{{\omega_0}^2-\omega^2}$$

Examples

FT. 1.

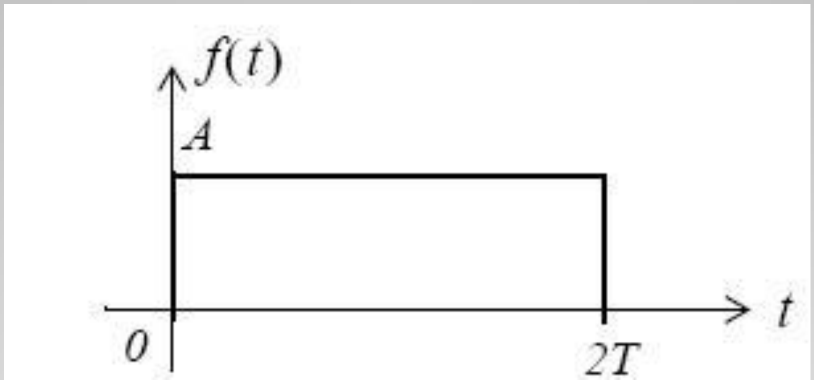
$$\mathbf{f(t) = A[u_0(t + T) - u_0(t - T)]}$$

$$\mathbf{A[u_0(t + T) - u_0(t - T)] \Leftrightarrow 2AT\frac{\sin\omega T}{\omega T}}$$



FT. 2.

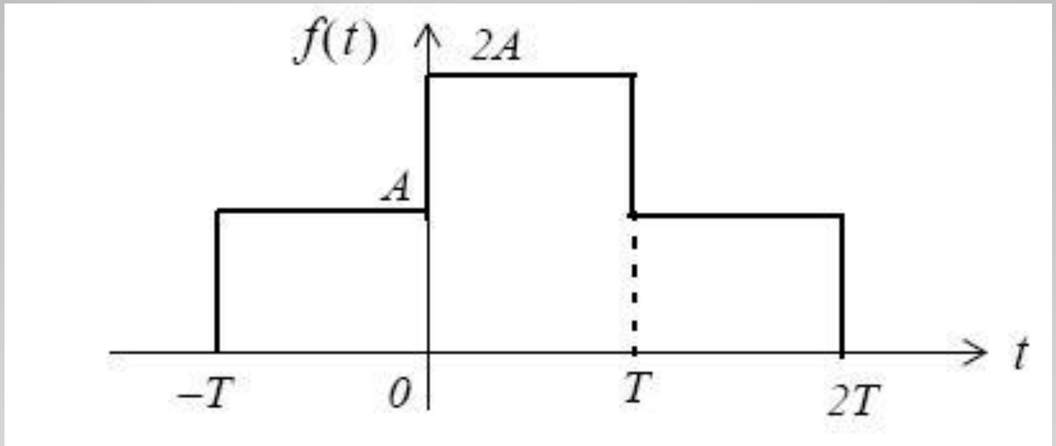
$$f(t) = A[u_0(t) - u_0(t - 2T)]$$



$$F(\omega) = 2ATe^{-j\omega T}\left(\frac{\sin\omega T}{\omega T}\right)$$

FT. 3.

$$f(t) = A[u_0(t+T) + u_0(t) - u_0(t-T) - u_0(t-2T)]$$



$$F(\omega) = 4ATe^{-j\frac{\omega T}{2}} \cos\left(\frac{\omega T}{2}\right)\left(\frac{\sin\omega T}{\omega T}\right)$$

FT. 4.

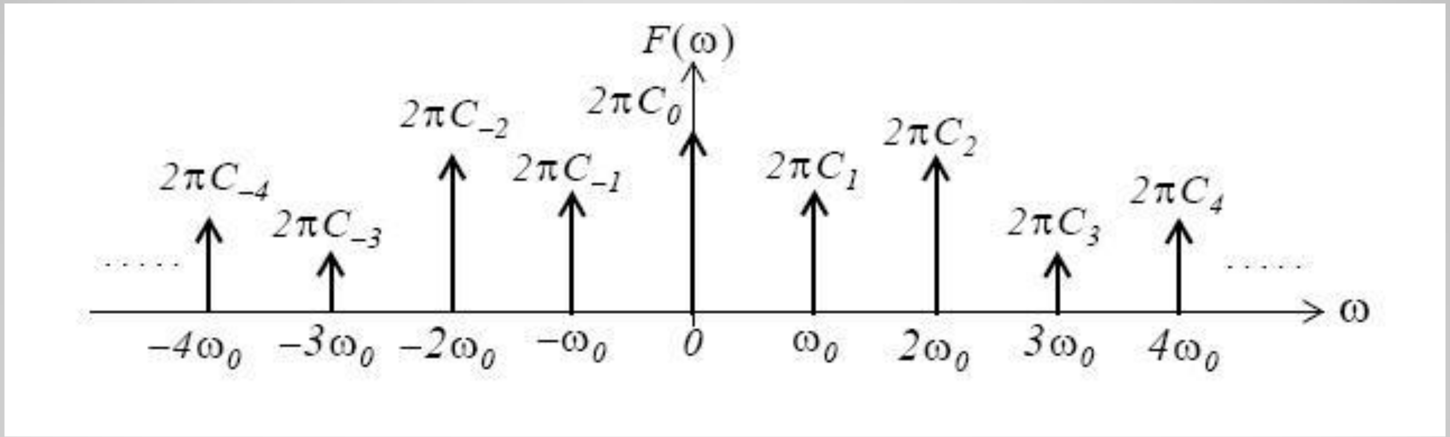
$$f(t) = A\cos(\omega_0 t)[u_0(t+T) - u_0(t-T)]$$

$$F(\omega) = AT\left[\frac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)T} + \frac{\sin(\omega+\omega_0)T}{(\omega+\omega_0)T}\right]$$

FT. 5.

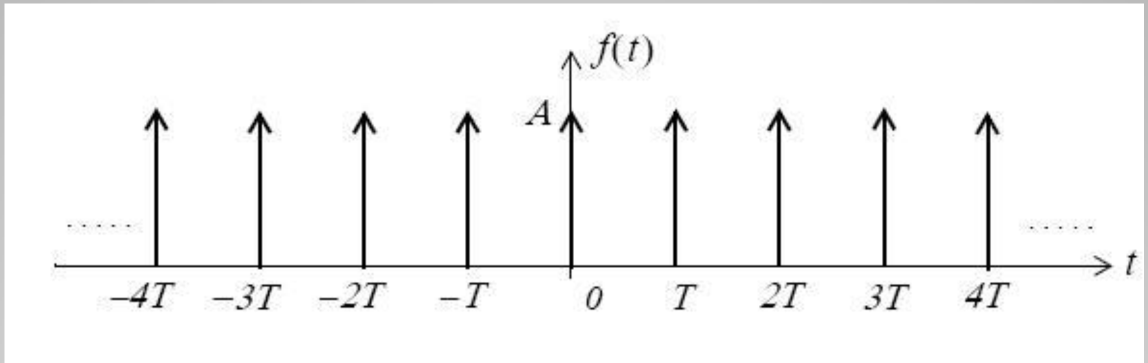
$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \, \delta(\omega - \omega_0)$$

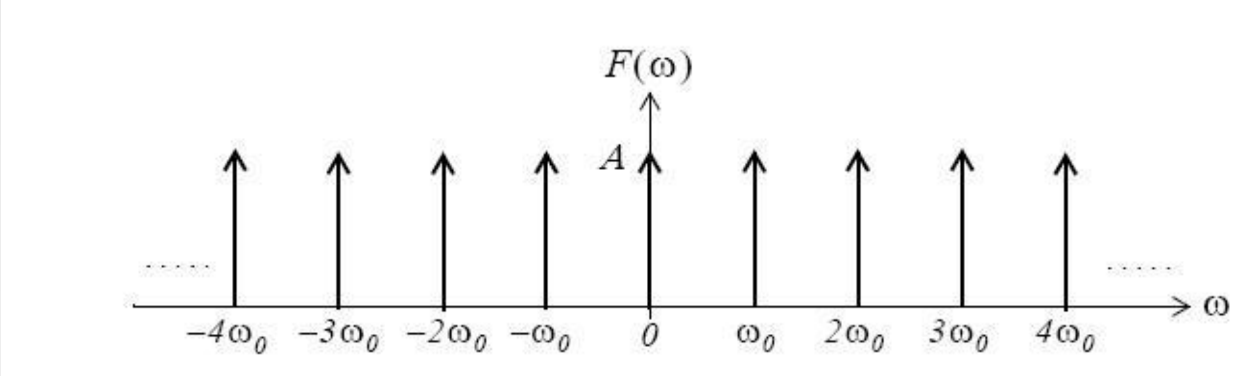


FT. 6.

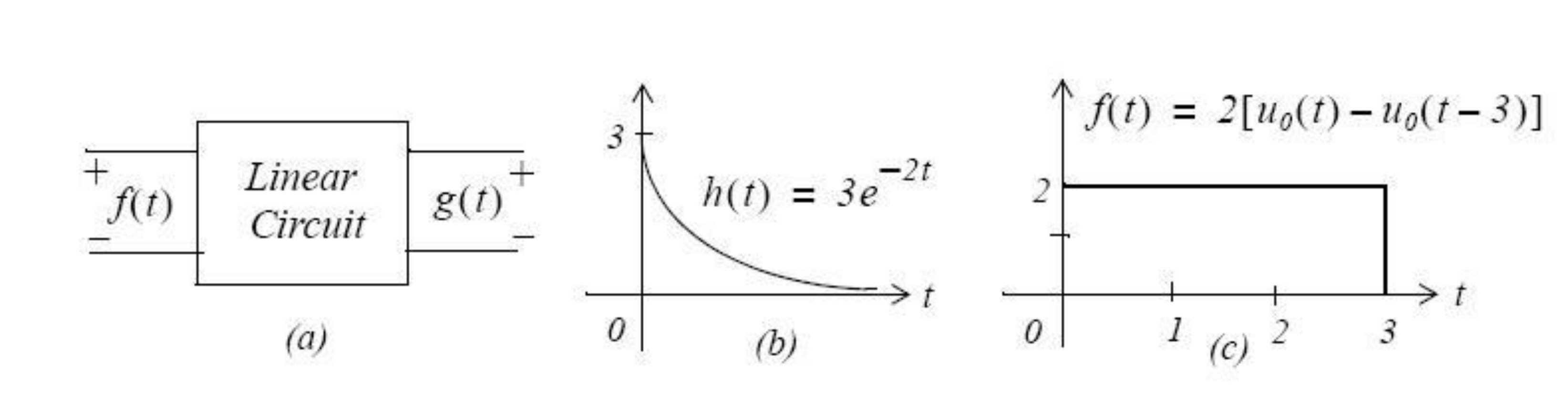
$$f(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$F(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

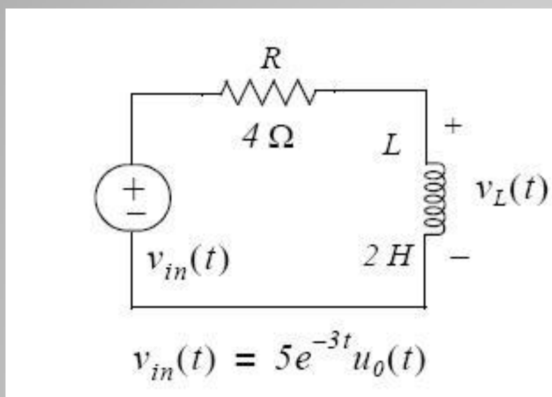


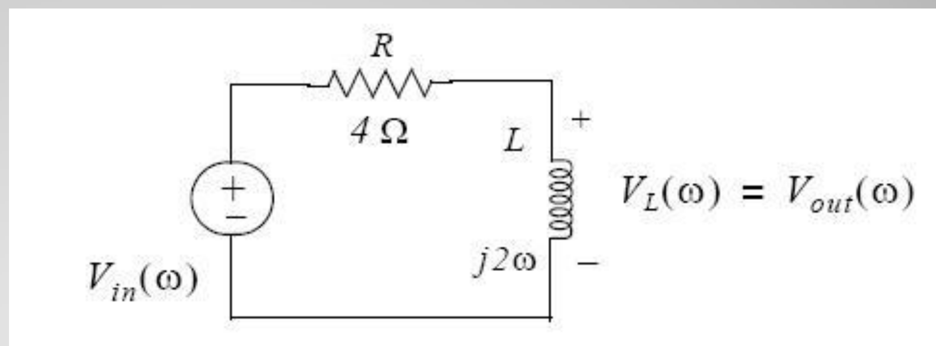
FT. 7. Use the Fourier transform method to compute the response g(t) when the input f(t) is as shown below.



$$g(t) = 1.5\{(1 - e^{-2t})u_0(t) - [1 - e^{-2(t-3)}]u_0(t-3)\}$$

FT. 8. For the curcuit in below, use the Fourier transform method, and the system function H(ω) to compute $v_L(t)$. Assume that $i_L(0_-)$.



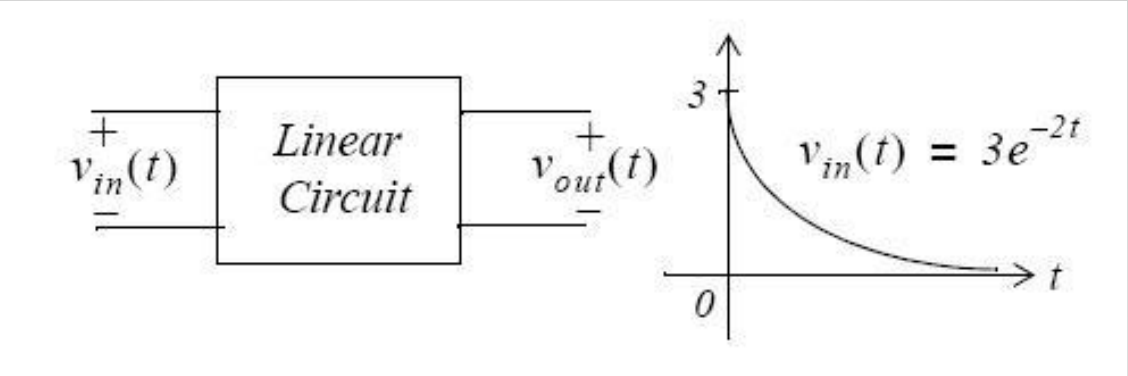


$$v_L(t) = 5(3e^{-3t} - 2e^{-2t})u_0(t)$$

FT. 9. For the linear network given below, the input-output relationship is

$$\frac{dv_{out}(t)}{dt} + 4v_{out}(t) = 10v_{in}(t)$$

Use the Fourier transform method, and the system function H(ω) to compute the output $v_{out}(t)$.



FT. 10. The voltage across an 1 Ω resistor is known to be

$$v_R(t) = 3e^{-2t}u_o(t)$$

Compute the energy dissipated in this resistor for $0<t<\infty$, and verify the result by application of Parseval's theorem.