SAMPLING

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Band-Limited Signals

If the Fourier transform F(ω) of x(t) has finite support, this analog signal is band-limited.

- If x(t) is a continuous analog signal and absolutely summable, the it can be sampled at intervals T for producing the signal s(n)=x(nT).

$$S(\omega) = \sum_{n=-\infty}^{+\infty} s(n) e^{-j\omega n}$$

Then s(n) is expressed by the discrete time Fourier transform (DTFT) equation

$$s(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S(\omega) e^{j\omega n} d\omega$$

Similarly, x(t) is presented by the Fourier transform (FT) equation.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Because of s(n)=x(nT), another representation of s(n) is

$$s(n) = x(nT) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega nT} d\omega$$

Theorem (DTFT and FT);

Suppose that $x(t) \in L^1$ is an analog signal, and s(n)=x(nT). If $s(n) \in L^1$ so that the DTFT sum converges uniformly to S(ω),

$$S(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\frac{\omega + 2\pi k}{T}\right)$$

Theorem (Shannon-Nyquist Sampling Theorem);

Suppose that $x(t) \in L^1$ is an analog signal, and s(n)=x(nT). If $s(n) \in L^1$ so that the DTFT sum converges uniformly to S(ω), then x(t) may be recovered form the samples s(n) if

- x(t) is band-limited.
- The sampling frequency $F = T^{-1} > 2F_{max}$, where $|X(\omega)| = 0$ for $\omega > 2\pi F_{max}$.

Definition (Nyquist Rate): If a signal x(t) is band-limited, its Nyquist rate is F=2Fmax, where Fmax is the least upper bound of values ω, where $|X(\omega)| \neq 0$.

Reconstruction

- It assumes that perfect discrete samples are obtained in the sampling operation.
- The interpolating signals are not finitely supported.
- There are an infinite number of signals that must be summed to achieve perfect reconstruction.

Theorem (Shannon-Nyquist Interpolation Theorem);

Suppose that $\mathbf{x}(\mathbf{t}) \in \mathbf{L}^1$ is an analog signal, and s(n)=x(nT). If $\mathbf{s}(\mathbf{n}) \in \mathbf{L}^1$ so that the DTFT sum converges uniformly to S(ω). Also let 2Fmax<F=1/T, where Fmax is the maximum frequency component of x(t) in Hz. Then x(t) may be recovered from s(n) by the following sum.

$$\mathbf{x}(\mathbf{t}) = \sum_{\mathbf{k}=-\infty}^{+\infty} \mathbf{s}(\mathbf{n})\,\mathbf{sinc}\left(\frac{\pi \mathbf{t}}{\mathbf{T}} - \pi \mathbf{n}\right)$$

Uncertainity Principle

Definition (Time and Frequency Domain Locality): The time domain locality of a signal x(t) is

$$\Delta_t^2(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |x(t)|^2 t^2 dt$$

and its frequency domain locality is

$$\Delta_\omega^2(X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 \omega^2 d\omega$$

Theorem (Uncertainity Principle);

Suppose that x(t) is an analog signal, $\|x\|_2 = 1$,
and $x^2(t): t \to 0$ as $t \to \infty$.

$$\sqrt{\frac{\pi}{2}} \leq \Delta_t(t)\Delta_\omega(X)$$